

Quasi Normal Modes for New Type Black Holes in New Massive Gravity

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Abstract

We obtain the quasi-normal mode frequencies of scalar perturbation on new type black holes in three dimensional new massive gravity. In some special cases, the exact quasi-normal mode frequencies are obtained by solving scalar field equations exactly. On some parameter regions, the highly damped quasi-normal mode frequencies are obtained in an analytic form by the so-called Stokes line method. This study on quasi-normal modes sheds some light on the mysterious nature of these black holes. We also comment about AdS/CFT correspondence and the entropy/area spectrum for new type black holes.

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1 Introduction

Three dimensional massive gravity model introduced by Bergshoeff, Hohm and Townsend (BHT) leads to some renewed interests in three dimensional higher curvature gravity, since this so-called new massive gravity (NMG) is composed of the standard Einstein-Hilbert term with a specific combination of scalar curvature square term and Ricci tensor square one [1]~[6]. One of the original motivation of this theory is the non-linear completion of Fierz-Pauli massive graviton theory or the realization of the massive gravity on three dimensional space. Though NMG has different nature, for instance parity even rather than odd, from topologically massive gravity (TMG) [7][8], these theories share common features in some aspects. One of the most interesting common aspects in these theories is the existence of (warped) AdS black hole solutions [2][9]~[12]. Though the incompatibility of the unitarity of massive graviton modes and AdS black hole solutions is not resolved, it was suggested that the special case may exist as a consistent quantum gravity [13].

Aside from massive gravitons, AdS black hole solutions give us another motivation to study NMG in the viewpoint of the AdS/CFT correspondence, as was done extensively in TMG [21]~[30]. One direction along the AdS/CFT correspondence, *a la* holographic c-theorem, is the extension of NMG to even higher curvature gravities, named as extended NMG [31][32]. In these cases, various AdS black hole solutions are found and the central charges of their conjectural dual CFT are identified [34]~[44]. On the contrary, there are different aspects in NMG from TMG, one of which is the existence of the new type black holes discovered by BHT in NMG which we will name *new type black holes* in the following [2]. These black holes exist only for a specific combination of parameters in the NMG Lagrangian.

New type black holes are asymptotically AdS and contain two parameters even in the static case. From the standpoint of the usual Einstein-Hilbert gravity, the existence of an additional parameter is mysterious for static and spherically symmetric black holes (or axi-symmetric ones in the three dimensional case). Since NMG is a higher derivative theory, the existence of an additional parameter may be regarded as natural when one considers an initial value problem. Though the masses of these black holes are obtained as a conserved charges in the form of the combination of black hole parameters, and the validity of the formula is supported by the first law of black hole thermodynamics and also by the dynamical new type black hole solutions, it is still unknown what is the physical interpretation of additional parameter. Suppose that the additional parameter or another combination of two parameters different from mass combination is identified as a conserved charge, it is puzzling that the first law of black hole thermodynamics is already satisfied without a new conserved charge. Moreover, mass seems to be the unique conserved quantity for static new type black holes from the symmetry consideration.

If the additional parameter is not a conserved charge, it is desirable to study non-conserved quantities relevant to black hole physics. One of these non-conserved dissipating quantities in black hole physics is quasi-normal mode(QNM) [45]~[47](See references there in). Some information about black holes can be obtained by perturbing them and studying ensuing behaviors. Since black holes are not closed systems, there are dissipating modes when black holes are perturbed. The dissipating modes in black holes may be defined by modes such as only ingoing part at the horizon and only radiating one at the infinity. These modes with a specific boundary condition are called QNMs. The same term is used for modes of dissipating fields other than metric perturbation in

the black hole background. In this paper we study QNM frequencies, which are complex numbers, of scalar fields on the new type black holes.

There are several suggestions how to define QNMs on black holes. One of them is to take the boundary conditions such as ingoing modes only at the horizon and outgoing modes only at the spatial infinity with complex frequencies. In the asymptotic *AdS* case, which is relevant to us, the Dirichlet boundary condition instead of outgoing one may be taken at the spatial infinity because of the existence of blowing up potential [48]. QNMs mean the decay of modes for the time elapse. Because of the complex nature of QNM frequencies, ingoing/outgoing boundary conditions lead to exponentially blowing behaviors of modes at the horizon and the spatial infinity for fixed time. By separation of variables, the radial part of field equations is given in the form of

$$\left[-\frac{d^2}{dx^2} - \omega^2 + U(x) \right] \Phi(x) = 0.$$

In the viewpoint of the radial equations, ingoing/outgoing boundary conditions mean taking the exponentially blowing part while ignoring exponentially vanishing one. This process may be ambiguous since we should take care of exponentially small parts which are usually difficult to control in approximate solutions or asymptotic series solutions of differential equations, though one may not confront such difficulties in exact solutions.

To overcome the ambiguity in boundary conditions taken in the domain of real values, it was proposed to perform the analytic continuation of radial coordinate to the complex domain and to impose boundary conditions along Stokes lines [50]. Stokes lines are defined by the contour such that would-be vanishing or blowing terms are equally contributing.¹ In our notation these lines can be defined by $\Im m(\omega x) = 0$. On these lines the boundary conditions can be imposed unambiguously. It may be useful to note that one needs some additional information about ω actually to draw Stokes lines on the complex plane of the radial coordinate. Along with these boundary conditions, a method is developed to obtain QNM frequencies using Stokes lines. This method is basically monodromy approach to QNMs, while one needs some modifications in the asymptotic *AdS* case [49]~[51].

In the asymptotic *AdS* case Stokes lines are not closed and so open lines are used to match approximate solutions. This Stokes line method gives us approximate results since it uses matchings among the approximate solutions through the interpolation. However, some important outcome in QNM frequencies can be obtained. For instance, the asymptotic QNM frequencies can be reliably identified, which is important in the area spectrum and entropy quantization through Hod's conjecture [52][53]. In our case of new type black holes, one can see some clues, through QNM frequencies, about which combination of parameters different from mass is appropriate.

This paper is organized as follows. In section 2, we give a brief review on the properties of new type black holes in NMG to fix our conventions. In section 3, we solve the scalar field equations on new type black hole backgrounds and obtain exact QNMs in some special cases, which include results in the case of the Bañados-Teitelboim-Zanelli (BTZ) black holes [54]. In this section we have taken the simple form of boundary conditions for QNMs. In section 4, we obtain highly damped QNMs in some general case using Stokes line method. We conclude with summary and several

¹In mathematical literatures these contours are called anti-Stokes lines.

comments in the final section. In the appendixes, we present some mathematical formulae and numerical computation of QNM frequencies for some parameter domains.

2 New Type Black Holes in NMG

2.1 New massive gravity : brief review

After Bergshoeff, Hohm and Townsend have introduced a higher curvature theory (NMG) as the non-linear completion of Fierz-Pauli massive graviton theory in three dimensions [1]~[4], of which Lagrangian consists of the standard Einstein-Hilbert term and a specific combination of scalar curvature square term and Ricci tensor square one, this theory has drawn renewed interests in three dimensional gravity in various view points, one of which is along *AdS/CFT* correspondence. In particular, it was shown that the combination of curvature squared terms is consistent with and, in fact, determined by the holographic c-theorem, which is a specific incarnation of *AdS/CFT* correspondence [31]~[37]. This insight leads to the extensions of NMG to even higher curvature theories. Other directions of the exploration of NMG include various (charged) black hole solutions [2][9]~[12], supersymmetric extension [5][6], generalization by including the gravitational Chern-Simons term [40], the computation of correlations and anomaly via *AdS/CFT* correspondence [19], the appropriate extension of Gibbons-Hawking boundary term [39], the Hamiltonian analysis of NMG [55], etc.

In this paper, we consider the simplest version of NMG of which action is given by²

$$S = \frac{\eta}{2\kappa^2} \int d^3x \sqrt{-g} \left[\sigma R + \frac{2}{l^2} + \frac{1}{m^2} K \right], \quad (1)$$

where η and σ take 1 or -1 , and K is a specific combination of scalar curvature square and Ricci tensor square defined by

$$K = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2. \quad (2)$$

Our convention is such that m^2 is always positive but the cosmological constant l^2 has no such restriction. The equations of motion(EOM) of NMG are given by

$$\mathcal{E}_{\mu\nu} = \eta \left[\sigma G_{\mu\nu} - \frac{1}{l^2} g_{\mu\nu} + \frac{1}{2m^2} K_{\mu\nu} \right] = 0, \quad (3)$$

where

$$K_{\mu\nu} = g_{\mu\nu} \left(3R_{\alpha\beta} R^{\alpha\beta} - \frac{13}{8} R^2 \right) + \frac{9}{2} R R_{\mu\nu} - 8R_{\mu\alpha} R_{\nu}^{\alpha} + \frac{1}{2} \left(4\nabla^2 R_{\mu\nu} - \nabla_{\mu} \nabla_{\nu} R - g_{\mu\nu} \nabla^2 R \right). \quad (4)$$

There is an interesting relation between the scalar K and the tensor $K_{\mu\nu}$

$$g^{\mu\nu} K_{\mu\nu} = K. \quad (5)$$

²We have introduced η for the various sign choice of terms in the action.

Another useful form of the action which is equivalent to (1) is given by

$$S = \frac{\eta}{2\kappa^2} \int d^3x \sqrt{-g} \left[\sigma R + \frac{2}{l^2} + f^{\mu\nu} G_{\mu\nu} - \frac{1}{4} m^2 (f^{\mu\nu} f_{\mu\nu} - f^2) \right], \quad (6)$$

where $f_{\mu\nu}$ is an auxiliary symmetric tensor field related to ‘Shouten’ tensor [56]

$$f_{\mu\nu} = \frac{2}{m^2} S_{\mu\nu}, \quad S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \quad (7)$$

and its trace $f = g^{\mu\nu} f_{\mu\nu}$.

2.2 New type black holes

Among various black hole solutions known in NMG: BTZ black holes, Warped AdS_3 black holes, New type of black holes, Lifshitz type black holes, etc [1][9][57], our main interests in this paper are the so-called (static) new type of black holes which exist only when $\sigma = 1$ and $l^2 = 1/m^2$. These black hole solutions contain two parameters though the nature of these parameters is obscure at the present. One of the motivations for our study is to improve this situation. The metric of new type black holes is given by

$$ds^2 = L^2 \left[-(r^2 + br + c) dt^2 + \frac{dr^2}{(r^2 + br + c)} + r^2 d\phi^2 \right], \quad L^2 = \frac{1}{2m^2} = \frac{l^2}{2}, \quad (8)$$

with outer and inner horizons at $r_{\pm} = \frac{1}{2}(-b \pm \sqrt{b^2 - 4c})$. The scalar curvature of this metric is given by

$$R = -\frac{6}{L^2} - \frac{2b}{L^2 r},$$

which shows us that there is a curvature singularity at $r = 0$ when $b \neq 0$. New type black holes may be classified according to signs of parameters b and c . This classification may be tabulated by introducing a new parameter $q \equiv r_-/r_+$, as follows: The other sign combinations of b and c do not represent black holes.

$q \equiv r_-/r_+$	b and c	r_+ and r_-
$q = 1$	$b < 0, c > 0 (b^2 = 4c)$	$r_+ > 0, r_- > 0 (r_+ = r_-)$
$0 < q < 1$	$b < 0, c > 0$	$r_+ > 0, r_- > 0 (r_+ > r_-)$
$q = 0$	$b < 0, c = 0$	$r_+ > 0, r_- = 0$
$-1 < q < 0$	$b < 0, c < 0$	$r_+ > 0, r_- < 0 (r_+ > r_-)$
$q = -1$	$b = 0, c < 0$	$r_+ > 0, r_- < 0 (r_+ = -r_-)$
$q < -1$	$b > 0, c < 0$	$r_+ > 0, r_- < 0 (r_+ < r_-)$

Note that new type black holes for $q = -1$ are nothing but non-rotating BTZ black holes.

Now, let us list some quantities of new type black holes. Firstly, the Hawking temperature of new type black holes can be read from the surface gravity or the periodicity of Euclideanized action as

$$T_H = \frac{r_+ - r_-}{4\pi L} = \frac{\sqrt{b^2 - 4c}}{4\pi L}. \quad (9)$$

The mass of new type black holes is identified as a conserved charge by

$$M = \frac{b^2 - 4c}{16G}, \quad (10)$$

which is also justified by the dynamical approach or *AdS/CFT* correspondence [34][35][58]. Let us recall that the Bekenstein-Hawking-Wald entropy of new type black holes is different from area law. More explicitly, it is given by

$$S_{BHW} = \frac{A_H}{4G} \eta \left[\sigma + \frac{1}{4m^2} \left(R_t^t + R_r^r - 3R_\phi^\phi \right) \right]_{\text{at horizon}},$$

where A_H is the horizon area and R_ν^μ are Ricci tensors. By computing the entropy on the outer horizon, this entropy can be written as the difference between the area of the outer horizon and the one of the inner horizon as follows

$$S_{BHW} = \frac{1}{4G} (A_+ - A_-) = \frac{\pi L}{2G} (r_+ - r_-) = \frac{\pi L}{2G} \sqrt{b^2 - 4c}. \quad (11)$$

It is interesting to note that the higher curvature effect leads to the inner horizon dependence of the black hole entropy. The above quantities satisfy the simple form of the first law of black hole thermodynamics as $dM = T_H dS_{BHW}$, which is also consistent with the *AdS/CFT* correspondence.

Though some physical quantities are identified and thermodynamic properties are checked, the existence of two parameters is still mysterious. It is unclear what is the meaning of another combination of parameters different from the mass combination, or what combination at all one should take. This problem may be rephrased in another way as follows.

Usually higher derivative or curvature effects are regarded as introducing additional degrees of freedom. This is the reason that the would-be nondynamical three dimensional gravity allows propagating massive graviton modes and becomes more interesting. In the form of initial value problems of EOM, one needs more initial values to solve the problem with higher derivatives. In these viewpoints, the above black hole entropy formula is counterintuitive, since the positive inner horizon area means the reduction of the entropy even if it comes from higher curvature effects. More distinguishably, there is one parameter family of black hole like solutions, that is, the extremal ones ($r_+ = r_-$), which lead to zero entropy. Therefore, one may envisage the domain of parameters of positive inner horizon area as forbidden or at least as disconnected from the case of negative inner horizon area. In a later section, we will show some clues on such expectation.

In order to understand these peculiar properties of the new type black holes, it is desirable to investigate these black holes in more detail. In this paper we study scalar perturbations or scalar fields on these black hole backgrounds. Specifically, we focus on the QNMs associated with scalar fields. It turns out that QNMs give some clues about the nature of parameters.

3 Exact QNMs of New Type Black Holes

The EOM of scalar fields on new type black hole backgrounds is written as

$$\nabla^2 \Psi - m^2 \Psi = 0, \quad (12)$$

where m is the mass of the scalar field. To obtain the QNMs of scalar fields on the new type black hole background, we recall that the metric of this black hole is given by (8). For our convenience, let us take $L = 1$ in the following, which can be recovered by dimensional reasoning. By the separation of variables $\Psi = R(r)e^{i\omega t + i\mu\phi}$, one obtains the radial equation of the above EOM. Through the change of variable

$$z \equiv \frac{r - r_+}{r - r_-}, \quad (13)$$

it becomes

$$R''(z) + \left[\frac{1}{z} - \frac{1}{z-1} + \frac{1}{z-z_0} \right] R'(z) + \frac{1}{z(z-1)(z-z_0)} \left[\frac{(z-1)(z-z_0)}{z(z_0-1)^2} \frac{\omega^2}{r_-^2} - \frac{(z-z_0)}{(z-1)} m^2 - \frac{(z-1)}{(z-z_0)} \frac{\mu^2}{r_-^2} \right] R(z) = 0, \quad (14)$$

where $z_0 \equiv r_+/r_- = 1/q$ and $'$ denotes the derivative with respect to the variable z .

Defining $R(z) = z^\alpha(1-z)^\beta(z_0-z)^\gamma \mathcal{H}(z)$ and choosing parameters α , β and γ without loss of generality as

$$\alpha = \frac{i\omega}{(1-1/z_0)r_+}, \quad \beta = 1 - \sqrt{1+m^2} \quad \text{and} \quad \gamma = \frac{\mu}{r_+} \sqrt{z_0}, \quad (15)$$

one obtains a differential equation for $\mathcal{H}(z)$ in the form of

$$\mathcal{H}''(z) + \left[\frac{2\alpha+1}{z} + \frac{2\beta-1}{z-1} + \frac{2\gamma+1}{z-z_0} \right] \mathcal{H}'(z) + \left[z_0(\alpha+\beta-2\alpha\beta-\beta^2) - \alpha - \gamma - 2\alpha\gamma - \gamma^2 + \left\{ (\alpha+\beta+\gamma)^2 - \alpha^2 \right\} z \right] \frac{\mathcal{H}(z)}{z(z-1)(z-z_0)} = 0, \quad (16)$$

which turns out to be just a Heun's differential equation [59]. Let us recall that the Heun's differential equation in the standard form is given by

$$\mathcal{H}''(z) + \left[\frac{\nu}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-z_0} \right] \mathcal{H}'(z) + \frac{(\lambda\xi z - \eta)}{z(z-1)(z-z_0)} \mathcal{H}(z) = 0,$$

with the condition $\epsilon = \lambda + \xi - \nu - \delta + 1$. Now, one can see that (16) is a Heun's differential equation by identifying

$$\nu = 2\alpha + 1, \quad \delta = 2\beta - 1, \quad \epsilon = 2\gamma + 1, \quad (17)$$

$$\eta = z_0(2\alpha\beta + \beta^2 - \alpha - \beta) + \gamma^2 + 2\alpha\gamma + \alpha + \gamma, \quad (18)$$

$$\{\lambda, \xi\} = \{2\alpha + \beta + \gamma, \beta + \gamma\} \quad \text{or} \quad \{\beta + \gamma, 2\alpha + \beta + \gamma\}. \quad (19)$$

There exists a local solution of the differential equation (16) near $z = 0$ which is given by a combination of local Heun function $H(z)$'s. By imposing ingoing boundary condition at the outer horizon, one obtains the radial function as

$$R(z) = z^\alpha(1-z)^\beta(z_0-z)^\gamma \mathcal{H}(z) = C_1 z^\alpha(1-z)^\beta(z_0-z)^\gamma H(z_0, \eta, \lambda, \xi, \nu, \delta|z). \quad (20)$$

Note that as r goes to the horizon ($z \rightarrow 0$), $R(z)$ is reduced to z^α which represents the ingoing mode since $\alpha = \frac{i\omega}{(1-1/z_0)r_+}$.

Though solutions for the scalar field perturbation on new type black holes can be written as a local Heun function, one cannot extract useful information from it since Heun functions do not have a closed form in most case and their properties are not well understood, yet. Moreover, one should impose the boundary condition at the spatial infinity for the complete specification of QNMs. However, in the special cases of new type black holes, we can find QNMs by reducing Heun functions to more familiar ones.

First of all, when $q = -1$, new type black holes are just the non-rotating BTZ black holes and QNMs were already obtained in terms of hypergeometric functions, which can also be derived in our approach by the reduction of Heun functions to hypergeometric ones. When $q = 0$, the radial function becomes a (reduced) confluent Heun function. In this case QNMs can also be obtained exactly using the connection formula of the (reduced) confluent Heun functions. When $q = 1$, zero modes can be obtained. Now, let us elaborate on these things.

Firstly, let us consider the $q = -1$ case, i.e. $r_- = -r_+$. The radial function is given by

$$R(z) = C_1 z^\alpha (1-z)^\beta (-1-z)^\gamma H(-1, \eta, \lambda, \xi, 2\alpha+1, 2\beta-1|z), \quad (21)$$

where η is given by Eq. (18) while $\{\lambda, \xi\}$ is chosen as $\{2\alpha+\beta+\gamma, \beta+\gamma\}$ without loss of generality. It has been known that a transformation from the Heun function to the hypergeometric function exists when special relations among parameters are satisfied (see Appendix A) [60]. As one can check that parameters in the case of $z_0 = -1$ satisfy the conditions for the transformation, the above solution in terms of Heun function (21), can be written in terms of hypergeometric one. Using this transformation, the solution of the radial equation can be written in the form of

$$R(z) = C'_1 z^\alpha (1-z)^\beta (1+z)^{-2\alpha-\beta} {}_2F_1\left(\alpha + \frac{\beta+\gamma}{2}, \alpha + \frac{\beta-\gamma}{2}, 2\alpha+1 \middle| \frac{4z}{(z+1)^2}\right). \quad (22)$$

Inserting the values of α, β, γ with coordinate change to $w = 4z/(z+1)^2$, one can see that this is the same function already obtained for radial function on BTZ black holes [61]. Therefore, imposing the boundary condition (Dirichlet one) at the spatial infinity, QNM frequencies are obtained as follows:

$$\omega_{QNM} = \pm\mu + i2r_+ \left[n + \frac{1}{2}(1 + \sqrt{1+m^2}) \right], \quad (23)$$

where $n = 0, 1, 2, \dots$.

Secondly, we consider the case $q = 0$, i.e. $r_- = 0$. This case can be obtained by taking a limit $z_0 \rightarrow \infty$ with $r_- = r_+/z_0$ in the Eq.(14). One can observe that this is the limit for the (reduced) confluent Heun's differential equation [62]. Therefore, Eq.(16) is reduced to a (reduced) confluent Heun's differential equation

$$\mathcal{H}''(z) + \left[\frac{2\bar{\alpha}+1}{z} + \frac{2\beta-1}{z-1} \right] \mathcal{H}'(z) - \frac{(\bar{\gamma}^2 z - \bar{\eta})}{z(z-1)} \mathcal{H}(z) = 0, \quad (24)$$

with the identification among parameters

$$\bar{\alpha} \equiv \lim_{z_0 \rightarrow \infty} \alpha = i \frac{\omega}{r_+}, \quad \bar{\gamma}^2 \equiv \lim_{z_0 \rightarrow \infty} \frac{\gamma^2}{z_0} = \frac{\mu^2}{r_+^2}, \quad \bar{\eta} \equiv \lim_{z_0 \rightarrow \infty} \frac{\eta}{z_0} = \bar{\gamma}^2 - \bar{\alpha} - \beta + 2\bar{\alpha}\beta + \beta^2. \quad (25)$$

The local solution of this equation near $z = 0$ is given by a linear combination of the (reduced) confluent Heun function $H_C(z)$'s. Imposing ingoing boundary condition at the outer horizon, the radial function can be represented by

$$R(z) = C_2 z^{\bar{\alpha}} (1-z)^{\beta} H_C \left(0, 2\bar{\alpha}, 2\beta - 2, -\bar{\gamma}^2, (\beta - 1)^2 + \bar{\gamma}^2 \middle| z \right). \quad (26)$$

In order to investigate the behavior at infinity, we can use the connection formula (see Appendix A) and obtain relevant part of the radial function for imposing the boundary condition at the infinity as

$$R(z) = C_2 \frac{\Gamma(2\bar{\alpha} + 1)\Gamma(2 - 2\beta)}{\Gamma(3 - 2\beta + \zeta)\Gamma(2\bar{\alpha} - \zeta)} z^{\bar{\alpha}} (1-z)^{1-\beta} H_C \left(0, 2\beta - 2, 2\bar{\alpha}, \bar{\gamma}^2, (\beta - 1)^2 \middle| 1 - z \right), \quad (27)$$

where ζ is determined by

$$\zeta^2 + (3 - 2\bar{\alpha} - 2\beta)\zeta + \bar{\gamma}^2 + \beta^2 - 3\bar{\alpha} + 2\bar{\alpha}\beta - 3\beta + 2 = 0. \quad (28)$$

The Dirichlet boundary condition at the spatial infinity ($z = 1$) requires that the coefficient given by gamma functions vanish. This gives the following QNM frequencies:

$$\omega_{QNM} = ir_+ \left[\frac{\frac{\mu^2}{2r_+^2} + (n + \sqrt{1 + m^2})(n + \sqrt{1 + m^2} + 1)}{2(n + \sqrt{1 + m^2}) + 1} \right]. \quad (29)$$

When n is very large, the asymptotic form of QNM frequencies is given by

$$\omega_{QNM} \sim i \frac{r_+}{2} n. \quad (30)$$

Finally, let us consider the extremal case $q = 1$, i.e. $r_- = r_+$ with denoting the horizon radius as $r_0 = r_- = r_+$. In this case the change of variable in the Eq.(13) is meaningless. Instead, let us set the variable, $z \equiv \frac{r-r_0}{r}$ and denote $R(z) = z^\alpha (1-z)^\beta \mathcal{G}(z)$. Then, the radial equation in terms of $\mathcal{G}(z)$ can be written in the form of

$$\mathcal{G}''(z) + \left(\frac{2(\alpha + 1)}{z} + \frac{1 - 2\beta}{1 - z} \right) \mathcal{G}'(z) + \frac{1}{z(1 - z)} \left[\left(\frac{1 - z}{z^3} \right) \frac{\omega^2}{r_0^2} + \frac{\mu^2}{r_0^2} - \alpha^2 - \beta^2 - 2\alpha\beta \right] \mathcal{G}(z) = 0, \quad (31)$$

by choosing parameters as

$$\alpha = -\frac{r_0 \pm \sqrt{r_0^2 + 4\mu^2 + 4m^2 r_0^2}}{2r_0}, \quad \beta = 1 \pm \sqrt{1 + m^2}.$$

When $\omega = 0$, it becomes the hypergeometric differential equation. In this case, one obtains the solution as

$$\begin{aligned} R(z) = & c_1 z^\alpha (1-z)^\beta {}_2F_1 \left(-\frac{\mu}{r_0} + \alpha + \beta, \frac{\mu}{r_0} + \alpha + \beta, 2\alpha + 2 \middle| z \right) \\ & + c_2 z^{-\alpha-1} (1-z)^\beta {}_2F_1 \left(-\frac{\mu}{r_0} - \alpha + \beta - 1, \frac{\mu}{r_0} - \alpha + \beta - 1, -2\alpha \middle| z \right). \end{aligned} \quad (32)$$

Considering another choice of change of variable and an appropriate ansatz for radial function, e.g. $z' = \frac{r}{r-r_0}$ and $R(z') = z'^{\alpha'} (1-z')^{\beta'} e^{xz'} G(z')$, the radial equation can be reduced to a

confluent Heun's differential equation³. Because of the absence for the connection formula among the confluent Heun functions at singularities, it is hard to find exact QNMs for this case. On the other hand, there were some other attempts to find QNMs analytically by perturbation method for Heun's differential equation [64][65].

4 Asymptotic QNMs in New Type Black Holes

In this section, we present the asymptotic QNM frequencies of scalar fields on new type black holes. To obtain these, we use the so-called Stokes line method which matches the approximate solutions of the radial equation by the analytic continuation of the radial coordinate to the complex plane [49]. It is sufficient to consider the asymptotic solutions near the infinity, the origin and the event horizon. After the approximate solutions in each relevant region are obtained, they are matched with appropriate boundary conditions. Through these matchings, one can read off the QNM frequencies of scalar fields on new type black holes.

Firstly, we need to solve the scalar perturbation equation (12) in each region. For this purpose it is convenient to introduce the tortoise coordinates x as follows

$$\frac{dx}{dr} = \frac{1}{(r - r_+)(r - r_-)}. \quad (33)$$

By choosing the integration constant such as $x(r = 0) = 0$, one obtains

$$x = \frac{1}{r_+ - r_-} \ln \frac{r - r_+}{r - r_-} + x_0, \quad (34)$$

where x_0 is given by

$$x_0 \equiv \frac{1}{r_+ - r_-} \ln \left(\frac{r_-}{r_+} \right) = \frac{1}{2\kappa_+} \ln \left(\frac{r_-}{r_+} \right). \quad (35)$$

It is interesting to note that r_+ should always be taken as a positive number to form black holes whereas r_- is allowed to be any value less than or equal to r_+ . When $r_- \leq 0$, there is just one real horizon. In the viewpoint of asymptotic QNM computation, $r_- = 0$ is a special case and should be treated separately, whereas the case of $r_- < 0$ can be done uniformly. In each region, the tortoise coordinate x behaves like

$$x \simeq \begin{cases} \frac{r}{r_+ r_-} & r \rightarrow 0 \\ -\infty & r \rightarrow r_+ \\ x_0 - \frac{1}{r} & r \rightarrow \infty \end{cases}.$$

By denoting $\Phi(x(r)) \equiv \sqrt{r}R(r)$, the radial equation of scalar field equation can be written as

$$\left[-\frac{d^2}{dx^2} - \omega^2 + U(x) \right] \Phi(x) = 0, \quad (36)$$

where the potential term is given by

$$U(x) = \left(1 - \frac{r_+}{r} \right) \left(1 - \frac{r_-}{r} \right) \left[\mu^2 + m^2 r^2 + \frac{1}{4} \left(3r^2 - (r_+ + r_-)r - r_+ r_- \right) \right]. \quad (37)$$

³It is equivalent to the generalized spheroidal wave equation [63].

The potential term in each region becomes

$$U(x) \simeq \begin{cases} \frac{r_+ r_-}{r^2} \left(\mu^2 - \frac{r_+ r_-}{4} \right) = \frac{j^2 - 1}{4x^2}, & j^2 \equiv \frac{4\mu^2}{r_+ r_-} & r \rightarrow 0 \quad (x \rightarrow 0) \\ 0 & & r \rightarrow r_+ \quad (x \rightarrow -\infty) \\ \left(\frac{3}{4} + m^2 \right) r^2 = \frac{j_\infty^2 - 1}{4(x - x_0)^2}, & j_\infty \equiv 2\sqrt{1 + m^2} & r \rightarrow \infty \quad (x \rightarrow x_0) \end{cases}.$$

In each singular point region, one can obtain the radial function, $\Phi(x)$, as follows

$$\Phi(x) = \begin{cases} A_+ \sqrt{2\pi\omega x} J_{\frac{j}{2}}(\omega x) + A_- \sqrt{2\pi\omega x} J_{-\frac{j}{2}}(\omega x), & r \rightarrow 0 \quad (x \rightarrow 0) \\ C_+ \sqrt{2\pi\omega(x_0 - x)} J_{\frac{j_\infty}{2}}(\omega(x_0 - x)) \\ \quad + C_- \sqrt{2\pi\omega(x_0 - x)} J_{-\frac{j_\infty}{2}}(\omega(x_0 - x)), & r \rightarrow \infty \quad (x \rightarrow x_0) \\ D_+ e^{i\omega x} + D_- e^{-i\omega x}, & r \rightarrow r_+ \quad (x \rightarrow -\infty) \end{cases} \quad (38)$$

where A_\pm, C_\pm and D_\pm are complex constants and $J_{\pm \frac{j}{2}}(\omega x)$ and $J_{\pm \frac{j_\infty}{2}}(\omega(x_0 - x))$ represent first kind Bessel functions. The Dirichlet boundary condition near the spatial infinity implies that this radial function should vanish there. Therefore we should take $C_- = 0$. The condition of $D_- = 0$ comes from the boundary condition for QNMs near the event horizon r_+ : $\Phi(x)$ should contain only ingoing mode $e^{i\omega x}$ there.

In order to obtain asymptotic QNM frequencies, let us examine the Stokes line which is defined by the curve $\Im m(\omega x) = 0$. For the asymptotic AdS_D solutions ($D \geq 4$), it has been known that the magnitudes of both $\Im m(\omega x)$ and $\Re e(\omega x)$ are comparable in the asymptotic QNM frequencies, which is related to the fact that ωx is asymptotically real when x goes to x_0 , i.e. $r \rightarrow \infty$. Along this Stokes line, modes will be oscillatory without any exponentially growing/decaying modes. In a neighborhood of the origin, the relation $x \sim \frac{r}{r_+ r_-}$ shows us that

$$r = \rho e^{i(n\pi + \theta_0)},$$

where $\rho > 0$ and $n = 0, 1$. It describes two lines emanating from the origin, spaced by an angle π . Note that the sign of ωx on these lines is alternatively positive and negative. The angle θ_0 can be obtained by

$$\theta_0 = \arg(x_0) = \begin{cases} 0 & \text{for } r_- > 0, \\ \tan^{-1} \left(\frac{\pi}{\ln |\frac{r_-}{r_+}|} \right) & \text{for } r_- < 0. \end{cases} \quad (39)$$

In the case of $r_- < 0$, the branch of the Stokes line emanating from the origin with the angle θ_0 goes to the infinity with positive ωx value. The other branch runs to the horizon with negative ωx value starting from the origin with its angle $\theta_0 - \pi$. The educated guess leads to Stokes line depicted in the Figure 1, which results in QNM frequencies consistent with numerical ones given in the Appendix B. In $r_- > 0$ case, the Stokes lines seems to be laid on the real axis. This lines do not seem to make sense in matching solutions around singular points. Therefore, we confine ourselves to the parameter domain $r_- < 0$ case only.

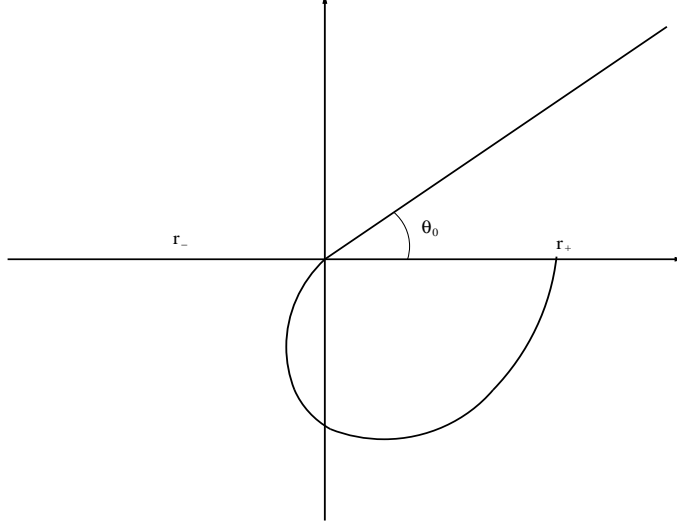


Figure 1: Stokes line

Now, we need to match the above solutions along the stokes line on the complex r plane. Using the asymptotic expansion of the first kind Bessel function in the region of $|\omega x| \rightarrow \infty$ [66], one obtains the expansions of the first and second solutions of Eq.(38) at some point on the positive branch, respectively as

$$\Phi_o(x) \sim e^{-i\frac{\pi}{4}}(A_+e^{-i\frac{j\pi}{4}} + A_-e^{i\frac{j\pi}{4}})e^{i\omega x} + e^{i\frac{\pi}{4}}(A_+e^{i\frac{j\pi}{4}} + A_-e^{-i\frac{j\pi}{4}})e^{-i\omega x}, \quad (40)$$

$$\Phi_\infty(x) \sim e^{i\frac{\pi}{4}}(C_+e^{-i\omega x_0}e^{i\frac{j\infty\pi}{4}})e^{i\omega x} + e^{-i\frac{\pi}{4}}(C_+e^{i\omega x_0}e^{-i\frac{j\infty\pi}{4}})e^{-i\omega x}. \quad (41)$$

Matching these two functions gives us a relation

$$A_+ \cos \left[\omega x_0 - \frac{(j_\infty + j)\pi}{4} \right] + A_- \cos \left[\omega x_0 - \frac{(j_\infty - j)\pi}{4} \right] = 0. \quad (42)$$

The solution near the origin (40) is on the positive branch of the Stokes line but the solution near the horizon is on the negative one. In order to match the near-horizon solution with the near-horizon one, one performs the rotation $x' = e^{-i\pi}x$ in the neighborhood of the origin while evading the singular origin. By doing this rotation near the origin and then using asymptotic expansion for the first solution of Eq.(38), the solution $\Phi_o(x)$ on the negative branch is represented by

$$\Phi_o(x) \sim e^{-i\frac{\pi}{4}}(A_+e^{-i\frac{j\pi}{4}} + A_-e^{i\frac{j\pi}{4}})e^{i\omega x} + e^{-i\frac{3\pi}{4}}(A_+e^{-i\frac{3j\pi}{4}} + A_-e^{i\frac{3j\pi}{4}})e^{-i\omega x}. \quad (43)$$

By matching the above solution with the solution near the event horizon given in the Eq.(38) one can see that $\Phi(x)$ should be given in the form of $e^{i\omega x}$. Consequently one obtains another relation

$$A_+e^{-i\frac{3j\pi}{4}} + A_-e^{i\frac{3j\pi}{4}} = 0. \quad (44)$$

The existence of nontrivial solutions of two relations (42) and (44) on the coefficients A_+ and A_- implies that their determinant should vanish. As a result, one obtains

$$\omega x_0 = \left(n + \frac{1}{2} + \frac{j_\infty}{4} \right) \pi - \frac{i}{2} \ln \left[2 \cos \left(\frac{j\pi}{2} \right) \right]. \quad (45)$$

Note that q is negative since we are dealing with the case $r_- < 0$.

Since we know the exact form of x_0 given in the Eq. (35), we can obtain the QNM frequencies analytically. Substituting $\kappa_+ = (1 - q)r_+/2 = 2\pi T_H$ into Eq. (35), we obtain

$$x_0 = \frac{1}{4\pi T_H} \ln(q) = \frac{1}{4T_H} \left(i + \frac{1}{\pi} \ln |q| \right). \quad (46)$$

Rewriting j in terms of q as

$$j = i \frac{1 - q}{\sqrt{|q|}} \frac{\mu}{2\pi T_H}. \quad (47)$$

QNM frequencies of scalar perturbation on new type black holes can be written as

$$\begin{aligned} \omega_{QNM} = & i \frac{4\pi^3 T_H}{\pi^2 + (\ln |q|)^2} \left(n + \frac{1}{2} + \frac{1}{2} \sqrt{1 + m^2} \right) + i \frac{2\pi T_H \ln |q|}{\pi^2 + (\ln |q|)^2} \ln \left[2 \cosh \left(\frac{1 - q}{\sqrt{|q|}} \frac{\mu}{4T_H} \right) \right] \\ & - \frac{4\pi^2 T_H \ln |q|}{\pi^2 + (\ln |q|)^2} \left(n + \frac{1}{2} + \frac{1}{2} \sqrt{1 + m^2} \right) + \frac{2\pi^2 T_H}{\pi^2 + (\ln |q|)^2} \ln \left[2 \cosh \left(\frac{1 - q}{\sqrt{|q|}} \frac{\mu}{4T_H} \right) \right]. \end{aligned} \quad (48)$$

In general, the above QNM frequencies show us that the dependence of overtone number n appears in both real and imaginary parts. So $\Im m(\omega)$ and $\Re e(\omega)$ take comparably large values with large n . This is consistent with the numerical results in large $|r_-|$ value region, some of which are represented in the appendix B. From the above formula, one can see that the QNM frequencies are linear functions of temperature $T_H \sim \sqrt{M}$ for small μ values, which tells us some information of black holes: At least for small $|q|$, the overtone number dependence indicates that $M \sim (r_+ - r_-)^2$.

In the limit of the non rotating BTZ black hole case i.e. $q = -1$ QNM frequencies become the simple form of

$$\omega_{QNM} = i4\pi T_H \left(n + \frac{1}{2} + \frac{1}{2} \sqrt{1 + m^2} \right) + 2T_H \ln \left[2 \cosh \left(\frac{\mu}{2T_H} \right) \right]. \quad (49)$$

The exact result of QNM frequencies has appeared in [61], which is given by the Eq.(23) in the previous section. For the small value of μ and the large value of the overtone number n ($|\mu| \ll n$), the real part of the QNM frequencies from the Stokes line method approaches to $2T_H \ln 2$ which is much smaller than the imaginary part, $4\pi n T_H$. Therefore we may regard the QNM frequencies of the BTZ black hole as a nearly pure imaginary number which is consistent with the asymptotically reality condition ωx_0 for small μ and large n .

5 Summary and Conclusion

In this paper we obtained QNM frequencies of scalar perturbation on static new type black holes in NMG. Though scalar field equations were solved exactly in terms of Heun function which is the solution of differential equations with four regular singular points, one cannot obtain much information from this representation since most properties of Heun functions are yet unknown.

Specifically, the so-called connection formulae among singular points are essential for the purpose of obtaining QNM frequencies by imposing boundary conditions at different singular points. Nevertheless, in some special cases, for instance in the BTZ limit, one can reduce Heun functions to more tractable ones and then obtain exact QNM frequencies analytically. However, since such reduction does not happen in most cases, we have adopted the Stokes line method to obtain asymptotic QNM frequencies analytically, which is supported by numerical results for the parameter domain $-r_- > r_+$.

These results reconfirm the mass formula of new type black holes and give some clues about their additional parameter. A natural combination of black hole parameters appears in ω_{QNM} is the form of $Q \equiv \ln|q| = \ln|r_-/r_+|$. Moreover, our results for small $Q \equiv \ln|q|$ leads to $\omega_{QNM} \sim T_H n \sim n(r_+ - r_-)$, which implies $M \sim (r_+ - r_-)^2$, since the overtone number, n , dependence of the imaginary part of asymptotic QNM frequencies is usually given by the mass of black hole backgrounds, or Choptuik scaling parameter. This is consistent with results from other approaches. Though this does not reveal the nature of additional parameter completely, it gives some clues about its nature. As alluded in the introduction, we have showed that there seems disconnection between $r_- < 0$ and $r_- > 0$ during our analysis. That is to say, Stokes line method or numerical one breaks down when $r_- = 0$. More explicitly, two methods are inapplicable in this case because x_0 cannot be defined and the potential of the radial equation in Eddington coordinates is unbounded. This may not be just inadequacy of methods to the problem as was argued by the entropy reduction in the $r_- > 0$ domain.

There are still many issues to be addressed in the future. First of all, it is very desirable to study *AdS/CFT* correspondence for QNM frequencies in new type black holes. Since new type black holes have different metric fall-off tails at the spatial infinity from BTZ black holes, the dual CFT may be different. Nevertheless, it is still perplexing that there is an additional parameter in the dual CFT picture. Thermal two point functions of perturbation operators in CFT can be determined more or less uniquely only by conformal symmetry and depends only on the conformal weights, which have been identified as black hole mass or temperature. It is unclear how to identify the additional parameter in the dual CFT. Maybe new type black holes are not good quantum objects allowed in quantum gravity.

Secondly, though we presented exact QNM frequencies for $r_- = 0$ and $r_- = -r_+$, these cases should be studied more carefully, since the potential in tortoise coordinates can have negative values when $0 < -r_- < r_+$ or $r_- \geq 0$. This can be referred to as the necessity of further study about stability of new type black holes. During our analysis, we have assumed that new type black holes are stable under perturbations. However, this is unverified assumption, though it is very plausible that new type black holes are stable near the parameter region corresponding to BTZ black holes which are stable since those can be embedded as Bogomol'nyi-Prasad-Sommerfield (BPS) states in the supersymmetric extension of NMG.

Thirdly, it is worthwhile to consider the problem on the quantization of the black hole entropy with the asymptotic QNMs. There have been many investigations concerned with this [52][53][67]~[72]. According to the work of [71][72], we can calculate the entropy spectra of new type black holes from asymptotic QNMs (i.e, for large overtone number). When $q = -1$ and $q = 0$, the entropy spectrum seems to be equally spaced. However, when $q < 0 (q \neq -1)$, the spacing of the entropy spectrum depends on the q factor, even though the entropy spectrum is still equally spaced for fixed

q value. The different aspect of the entropy spectrum may be caused by the different behavior of the asymptotic QNMs with both highly oscillating and damping modes. Some similar phenomena are observed for large Schwarzschild-AdS black holes in $D \geq 4$. Further investigations are needed to understand the entropy spectrum using the asymptotic QNMs.

Finally, though numerical results support Stokes line or monodromy method in general, the rigorous mathematical justification of this method is still lacking. More concretely, it is unclear how to estimate errors in Stokes line method and how to systematically compute the next orders in this approach. Usual convergent or asymptotic series approach to QNM frequencies has definite error estimates and can be investigated systematically, which leads to numerical computation of QNM frequencies. Therefore, it is also very interesting to investigate Stokes line method rigorously and verify its validity in general.

Acknowledgements

S.H.Y was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) through the Center for Quantum Spacetime(CQUeST) of Sogang University with grant number 2005-0049409 and in part with the grant number 2009-0085995. S.H.Y would like to thank Prof. Seungjoon Hyun at Yonsei University for discussion some time ago. This work of S.N and J.D.P was supported by a grant from the Kyung Hee University in 2009(KHU-20110060). S.N and Y.K were supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(No.2010-0008109). S.N was also supported by the National Research Foundation of Korea(NRF) grant funded by the Korean government(MEST)(No. 2009-0063068).

A The reduction of the Heun function to the Hypergeometric function

In this appendix some mathematical identities are presented, which are useful in the main text.

A.1 Heun to hypergeometric

The reduction formulae from the Heun function to the Hypergeometric one by polynomial transformations are given in mathematical literatures, for instance see [60]. In summary, when four singular points, $(0, 1, z_0, \infty)$ of the Heun function satisfy harmonic or equianharmonic conditions there are polynomial transformation from Heun function to Hypergeometric one. The harmonic condition means the three points $(0, 1, z_0)$ are collinear and equally spaced, which is relevant to our case. In [60], $z_0 = 2$ is taken as the canonical value with the polynomial transformation $R(t) = t(2 - t)$, see the Eq. (3.5.a). The reduction formula in the case of $z_0 = -1$ can be obtained by choosing a Möbius transformation from z to t as

$$t \equiv \frac{2z}{z+1}.$$

Noting that $R(t) = t(2 - t) = \frac{4z}{(z+1)^2}$ and (3.5.a) actually means two equations, one can see that there are two possible ways to reduce Heun functions to Hypergeometric ones as follows. That is

$$H(-1, \eta, \lambda, \xi, \nu, \delta | z) = (z+1)^{-\alpha} {}_2F_1\left(\frac{\lambda}{2}, \frac{\nu + \delta - \xi}{2}, \gamma \middle| \frac{4z}{(z+1)^2}\right), \quad (50)$$

with relations among parameters as

$$\eta = \lambda(\xi - \delta), \quad \gamma = 1 + \lambda - \xi, \quad (51)$$

or the similar formula with λ and ξ interchanged. This formula explains the reduction of radial equation from new type black holes to BTZ ones when $r_- = -r_+$. Here, our convention of the Hypergeometric function is

$$0 = \left[\frac{d^2}{dw^2} + \left(\frac{c}{w} + \frac{1+a+b-c}{w-1} \right) \frac{d}{dw} - \frac{ab}{w(1-w)} \right] {}_2F_1(a, b, c | w). \quad (52)$$

A.2 Confluent Heun to hypergeometric

The (reduced) confluent Heun's differential equation is given by

$$\mathcal{H}''(z) + \left(\frac{b+1}{z} + \frac{c+1}{z-1} \right) \mathcal{H}'(z) + \frac{(dz - \epsilon)}{z(z-1)} \mathcal{H}(z) = 0. \quad (53)$$

The general solution of this equation near $z = 0$ is given by a linear combination of the (reduced) confluent Heun function $H_C(z)$'s

$$\mathcal{H}(z) = c_1 H_C(0, b, c, d, e | z) + c_2 z^{-b} H_C(0, -b, c, d, e | z) \quad (54)$$

where $e \equiv \frac{1}{2}\{1 - (b+1)(c+1) - 2\epsilon\}$.

There is a connection formula which is given by

$$\begin{aligned} H_C(0, b, c, d, e|z) &= c_1 \frac{\Gamma(b+1)\Gamma(-c)}{\Gamma(1-c+\zeta)\Gamma(b-\zeta)} H_C(0, c, b, -d, e+d|1-z) \\ &+ c_2 \frac{\Gamma(b+1)\Gamma(c)}{\Gamma(1+c+\sigma)\Gamma(b-\sigma)} (1-z)^{-c} H_C(0, -c, b, -d, e+d|1-z), \end{aligned} \quad (55)$$

where

$$\begin{aligned} \zeta^2 + (1-b-c)\zeta - \epsilon - b - c + \frac{d}{2} &= 0, \\ \sigma^2 + (1-b+c)\sigma - \epsilon - b(c+1) + \frac{d}{2} &= 0. \end{aligned}$$

The above formula means that the confluent Heun function about $z = 0$ can be connected with some combination of the two solutions about $z = 1$ by analytic continuation.

When $d = 0$, the reduction formula from the (reduced) confluent Heun function to the hypergeometric one is given as follows:

$$H_C(0, b, c, 0, e|z) = {}_2F_1(A, B, C|z), \quad (56)$$

where

$$\begin{aligned} \{A, B\} &\equiv \left\{ \frac{b+c \pm \sqrt{b^2+c^2-4e+1}+1}{2}, \frac{b+c \mp \sqrt{b^2+c^2-4e+1}+1}{2} \right\} \\ C &\equiv b+1. \end{aligned}$$

A.3 Hypergeometric function and connection formula

The standard form of the hypergeometric differential equation is given by

$$F''(z) + \frac{c - (a+b+1)z}{z(1-z)} F'(z) - \frac{ab}{z(1-z)} F(z) = 0. \quad (57)$$

The solutions near $z = 0$ is given by

$$F(z) = {}_2F_1(a, b, c|z) + z^{1-c} {}_2F_1(a-c+1, b-c+1, 2-c|z). \quad (58)$$

The connection formula from the hypergeometric function, ${}_2F_1(a, b, c|z)$, about $z = 0$ to another one about $z = 1$ is given by

$$\begin{aligned} {}_2F_1(a, b, c|z) &= (1-z)^{c-a-b} \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c-a, c-b, 1-a-b+c|1-z) \\ &+ \frac{\Gamma(c)\Gamma(-a-b+c)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, 1+a+b-c|1-z). \end{aligned} \quad (59)$$

B Numerical results

In this appendix, we present some details about the numerical computation of QNM frequencies following [48][74][75]. New type black holes with ingoing Eddington coordinates ($v \equiv t + x$) are given by

$$ds^2 = L^2 \left[-f(r)dv^2 + 2dvdr + r^2 d\phi^2 \right], \quad f(r) \equiv (r - r_+)(r - r_-). \quad (60)$$

Scalar fields on these black hole backgrounds may be decomposed as $\psi = \frac{1}{\sqrt{r}}\Phi(r) e^{i\omega v} e^{i\mu\phi}$ under the convention taken in this paper. Note that in this convention ω_{QNM} should have positive imaginary part. The above separation of variables leads to the radial equation of massless scalars as

$$f(r)\Phi''(r) + \left(f'(r) + 2i\omega\right)\Phi'(r) - V(r)\Phi(r) = 0, \quad ' \equiv \frac{d}{dr}, \quad (61)$$

where

$$\begin{aligned} V(r) &\equiv -\frac{1}{4r^2}f(r) + \frac{1}{2r}f'(r) + \frac{\mu^2}{r^2} \\ &= \frac{3}{4} - \frac{r_+ + r_-}{4r} + \frac{\mu^2 - r_+ r_-}{4r^2}. \end{aligned}$$

By introducing new variable $y \equiv 1/r$, one obtains

$$s(y)\frac{d^2}{dy^2}\Phi + \frac{t(y)}{y - y_+}\frac{d}{dy}\Phi + \frac{u(y)}{(y - y_+)^2}\Phi = 0,$$

where $y_+ \equiv 1/r_+$, $y_- \equiv 1/r_-$ and

$$\begin{aligned} s(y) &\equiv -\frac{y^2}{y_+ y_-}(y - y_-), \\ t(y) &\equiv -\frac{2}{y_+ y_-}y(y - y_+)(y - y_-) + 2y - \frac{y_+ + y_-}{y_+ y_-}y^2 + 2i\omega y^2, \\ u(y) &\equiv (y - y_+)V(r). \end{aligned}$$

Series solution ansatz is taken as

$$\Phi(y) = \sum_{\ell=0}^{\infty} a_{\ell}(y - y_+)^{\ell},$$

which gives us recursion relation of a_{ℓ} in terms of the coefficients of mode expansion of $s(y), t(y), u(y)$

$$\begin{aligned} a_{\ell} &= -\frac{1}{P_{\ell}} \sum_{k=0}^{\ell-1} \left[k(k-1)s_{\ell-k} + kt_{\ell-k} + u_{\ell-k} \right] a_k, \\ P_{\ell} &= \ell(\ell-1)s_0 + \ell t_0 = 2y_+^2 \ell(\ell\kappa_+ + i\omega). \end{aligned}$$

These formulae can be implemented easily in mathematica [76], which leads to the following results in the case of $-r_- \geq r_+$ as follows.

r_+	r_-	μ	ω_{QNM}
1	-10	0	18.0486 + 20.9188 i
1	-10	1	18.5137+ 20.5174 i
1	-10	2	19.5422+ 19.6755 i
1	-10	3	20.7523+ 18.7317 i
1	-10	4	22.0311+ 17.7672 i
1	-10	5	23.3505+ 16.7959 i

r_+	r_-	μ	ω_{QNM}
100	-200	0	108.272 + 284.046 i
100	-200	1	108.282+ 284.044 i
100	-200	2	108.312+ 284.036 i
100	-200	3	108.363 + 284.024 i
100	-200	4	108.433 + 284.006 i
100	-200	5	108.524 + 283.984 i

r_+	r_-	μ	ω_{QNM}
10	-20	0	10.8272 + 28.4046 i
10	-20	1	10.9274 + 28.3797 i
10	-20	2	11.2203 + 28.3069 i
10	-10	3	11.6854 + 28.1916 i
10	-10	4	12.2947 + 28.0413 i
10	-10	5	13.0192 + 27.8644 i

r_+	r_-	μ	ω_{QNM}
10	-10	0	0.00000+ 20.0044 i
10	-10	1	1.00022 + 20.0022 i
10	-10	2	2.00045+ 20.0022 i
10	-10	3	3.00068 + 20.0022 i
10	-10	4	4.00091 + 20.0022 i
10	-10	5	5.00113 + 20.0022 i

One can see that there are very small variations of the numerical results when μ values are changed, in particular for large r_+ and r_- . Of course, this numerical results are just taken for the low lying modes and the number of interaction are taken up to 50 times i.e. $\ell = 50$, but it strongly indicate the highly damped mode obtained by Stokes line methods is reliable at least $-r_- \geq r_+$.

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<http://www.phy.olemiss.edu/~berti/qnms.html>,
<http://gamow.ist.utl.pt/~vitor/ringdown.html>